Positivity of Toeplitz determinants formed by rising factorial series and properties of related polynomials

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Abstract. In this note we prove positivity of Maclaurin coefficients of polynomials written in terms of rising factorials and arbitrary log-concave sequences. These polynomials arise naturally when studying log-concavity of rising factorial series. We propose several conjectures concerning zeros and coefficients of a generalized form of those polynomials. We also consider polynomials whose generating functions are higher order Toeplitz determinants formed by rising factorial series. We make three conjectures about these polynomials. All proposed conjectures are supported by numerical evidence.

Keywords: *Log-concavity, Pólya frequency sequences, Toeplitz determinant, stability, hyperbolicity, rising factorial, hypergeometric functions*

MSC2010: 26C10, 05A20

1. Introduction. The confluent hypergeometric function is defined by the series

$$ {}_1F_1(a;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}, \tag{1} $$

where $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is rising factorial or Pochhammer symbol. It was proved by Barnard, Gordy and Richards in [BGR] that the function

$$ z \to \begin{vmatrix} {}_1F_1(a;c;z) & {}_1F_1(a+1;c;z) \\ {}_1F_1(a-1;c;z) & {}_1F_1(a;c;z) \end{vmatrix} $$

has positive Maclaurin coefficients if $a>0$, $c>-1$ $(c\neq 0)$. This has been extended by Karp and Sitnik in [KS] to the determinant $(\alpha,\beta>0)$

$$ z \to \begin{vmatrix} f(x+\alpha;z) & f(x+\alpha+\beta;z) \\ f(x;z) & f(x+\beta;z) \end{vmatrix}, \text{ where } f(x;z) := \sum_{n=0}^{\infty} f_k(x)_k\frac{z^k}{k!} \tag{2} $$

and $\{f_k\}_{k=0}^{\infty}$ is any non-negative log-concave sequence without internal zeros, i.e. $f_k^2 \geq f_{k-1}f_{k+1}$, $k=1,2,\ldots$, and if $f_N=0$ for some $N>0$ then $f_k=0$ for all $k\geq N$. Since

$$ \begin{vmatrix} f(x+\alpha;z) & f(x+\alpha+\beta;z) \\ f(x;z) & f(x+\beta;z) \end{vmatrix} = \sum_{n=2}^{\infty} Q_n^{\alpha,\beta}(x)\frac{z^n}{n!}, $$

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where

$$Q_n^{\alpha,\beta}(x) := \sum_{k=0}^{n} f_k f_{n-k} \binom{n}{k} \left[(x+\alpha)_k (x+\beta)_{n-k} - (x+\alpha+\beta)_k (x)_{n-k}\right], \tag{3}$$

Theorem 1 from [KS] can be restated as follows:

Theorem A *Suppose $\{f_k\}_{k=0}^n$ is a non-negative log-concave sequence without internal zeros, $\alpha, \beta > 0$, $n \geq 2$. Then $Q_n^{\alpha,\beta}(x) \geq 0$ for all $x \geq 0$. The inequality is strict unless $f_k = q^k$, $k = 0, 1, \ldots, n$ for some $q > 0$.*

Note that this theorem does not cover the above result from [BGR] completely since Theorem A requires x to be non-negative while the result in [BGR] is valid for $x = a - 1 > -1$. On several occasions (see, for instance, [KarpAIM, KarpCMFT] I proposed the following two conjectures:

Conjecture 1 *If $f_k^2 > f_{k-1}f_{k+1}$, $k = 1, 2, \ldots, n-1$, $n \geq 3$, then $Q_n^{\alpha,\beta}(x)$ has positive coefficients at x^j, $j = 0, 1, \ldots, n-2$*

Recall that a polynomial is called Hurwitz stable if all its zeros have negative real part. See details and extensions in [T].

Conjecture 2 *If $f_k^2 > f_{k-1}f_{k+1}$, $k = 1, 2, \ldots, n-1$, $n \geq 3$, then $Q_n^{\alpha,\beta}(x)$ is Hurwitz stable.*

For polynomials with real coefficients stability implies positivity of coefficients (this result is usually attributed to A. Stodola (1893)) so that Conjecture 1 is true if Conjecture 2 holds.

Let me also propose a third conjecture that has not been presented elsewhere. It requires the notion of Pólya frequency sequence defined formally in section 4 below. Briefly, $\{f_k\}_{k=0}^n$ is PF_∞ if all minors of the infinite matrix (11) are non-negative.

Conjecture 3 *If $\{f_k\}_{k=0}^n$ is PF_∞, $n \geq 3$, then all zeros of $Q_n^{1,1}(x-1)$ are real and negative.*

Let me remark that Conjecture 3 fails for $Q_n^{\alpha,\beta}(x)$ with arbitrary $\alpha, \beta > 0$ and so does it for $Q_n^{1,1}(x-1)$ when $\{f_k\}_{k=0}^n$ is only log-concave (PF_∞ is much stronger requirement than log-concavity, see details in section 4). I have explicit (but a bit cumbersome) counterexamples that demonstrate these claims. All three conjectures are supported by massive numerical evidence.

In a relatively recent work [IsmLaf] Ismail and Laforgia and, more recently, Baricz and Ismail [BarIsm] proved absolute or complete monotonicity of numerous Hankel determinants formed by special functions which possess the integral representation

$$f_n = \int_{\alpha}^{\beta} [\phi(t)]^n d\mu(t),$$

where both the function ϕ and the measure μ may depend on parameters. When the size of the determinant is equal to 2 their results reduce to positivity and integral representations for $f_n f_{n+2} - f_{n+1}^2$. The positivity of this expression is discrete log-convexity of (or reverse Turán type inequality for) f_n. Unfortunately, the technique used in these papers does not extend to log-concavity (discrete or not) as far as I can see, although some discrete log-concavity results are proved in [BarIsm] employing a different method.

The purpose of this note is twofold. First, we prove the positivity of the coefficients of $Q_n^{1,1}(x-1)$ settling a particular case of Conjecture 1. This furnishes a far-reaching extension of the result of [BGR] and partially of [KS]. Second, we consider a higher order Toeplitz determinant whose entries are functions defined in (2). We give power series expansion of such determinant in powers of z with coefficients being polynomials in x. We make several conjectures about these polynomials serving as natural generalizations of Conjectures 1 and 3 for $Q_n^{1,1}(x-1)$.

2. Preliminaries. We will need several lemmas which we present in this section. We will always assume that the sequence $\{f_k\}$ is not a zero sequence.

Lemma 1 *Suppose $\{f_k\}_{k=0}^n$ has no internal zeros and $f_k^2 \geq f_{k-1}f_{k+1}$, $k = 1, 2, \dots, n-1$. If the real sequence $M_0, M_1, \dots, M_{[n/2]}$ satisfying $M_{[n/2]} > 0$ and $\sum_{k=0}^{[n/2]} M_k \geq 0$ has one change of sign, then*

$$\sum_{0 \leq k \leq n/2} f_k f_{n-k} M_k \geq 0. \tag{4}$$

Equality is only attained if $f_k = \alpha^k$, $\alpha > 0$, and $\sum_{k=0}^{[n/2]} M_k = 0$.

Proof. Suppose $f_k > 0$, $k = s, \dots, p$, $s \geq 0$, $p \leq n$. Log-concavity of $\{f_k\}_{k=0}^n$ clearly implies that $\{f_k/f_{k-1}\}_{k=s+1}^p$ is decreasing, so that for $s+1 \leq k \leq n-k+1 \leq p+1$

$$\frac{f_k}{f_{k-1}} \geq \frac{f_{n-k+1}}{f_{n-k}} \ \Leftrightarrow \ f_k f_{n-k} \geq f_{k-1} f_{n-k+1}.$$

Since $k \leq n-k+1$ is true for all $k = 1, 2, \dots, [n/2]$, the weights $f_k f_{n-k}$ assigned to negative M_ks in (4) are smaller than those assigned to positive M_ks leading to (4). The equality statement is obvious. $\square$

We will use the formula

$$\prod_{k=1}^{q} (x + a_k) = \sum_{k=0}^{q} e_{q-k}(a_1, \dots, a_q) x^k, \tag{5}$$

where $e_m(a_1, \dots, a_q)$ denotes m-th elementary symmetric polynomial,

$$e_k(a_1, \dots, a_q) = \sum_{1 \leq j_1 < j_2 \cdots < j_k \leq q} a_{j_1} a_{j_2} \cdots a_{j_k}.$$

The key fact about elementary symmetric polynomials that we will need requires the notion of majorization [MOA, Definition A.2, formula (12)]. It is said that $B = (b_1, \dots, b_q)$ is weakly super-majorized by $A = (a_1, \dots, a_q)$ (symbolized by $B \prec^W A$) if

$$\begin{aligned} &0 < a_1 \leq a_2 \leq \cdots \leq a_q, \quad 0 < b_1 \leq b_2 \leq \cdots \leq b_q, \\ &\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for} \quad k = 1, 2 \dots, q. \end{aligned} \tag{6}$$

Lemma 2 *Suppose $B \prec^W A$. Then*

$$\frac{e_k(a_1, \dots, a_q)}{e_{k-1}(a_1, \dots, a_q)} \leq \frac{e_k(b_1, \dots, b_q)}{e_{k-1}(b_1, \dots, b_q)}, \quad k = 1, 2, \dots, q.$$

Proof. According to [MOA, 3.A.8] $B \prec^W A$ implies that $\phi(A) \leq \phi(B)$ if and only if $\phi(x)$ is Schur-concave and increasing in each variable. Hence, we should choose

$$\phi_k(x_1, \dots, x_q) = \frac{e_k(x_1, \dots, x_q)}{e_{k-1}(x_1, \dots, x_q)}, \quad k = 1, 2, \dots, q.$$

Schur-concavity of these functions has been proved by Schur (1923) - see [MOA, 3.F.3]. It is left to show that ϕ_k is increasing in each variable. Due to symmetry we can take x_1 to be variable

thinking of $x_2, \ldots, x_q$ as being fixed. Using the definition of elementary symmetric polynomials we see that for $k \geq 2$

$$\phi_k(x_1, \ldots, x_q) = \frac{x_1 e_{k-1}(x_2, \ldots, x_q) + e_k(x_2, \ldots, x_q)}{x_1 e_{k-2}(x_2, \ldots, x_q) + e_{k-1}(x_2, \ldots, x_q)}.$$

So taking derivative with respect to x_1 we obtain ($e_m = e_m(x_2, \ldots, x_q)$ for brevity):

$$\frac{\partial \phi_k(x_1, \ldots, x_q)}{\partial x_1} = \frac{e_{k-1}(x_1 e_{k-2} + e_{k-1}) - e_{k-2}(x_1 e_{k-1} + e_k)}{[x_1 e_{k-2} + e_{k-1}]^2} = \frac{e_{k-1}^2 - e_k e_{k-2}}{[x_1 e_{k-2} + e_{k-1}]^2} \geq 0.$$

Non-negativity holds by Newton's inequalities. □

Next lemma is a part of Theorem A.

Lemma 3 *Suppose $f_k = 1$ for all $k = 0, 1, \ldots, n$. Then $Q_n^{\alpha,\beta}(x) \equiv 0$.*

Proof. If $f_k = 1$ for all $k = 0, 1, \ldots, n$, then $Q_n^{\alpha,\beta}(x)/n!$ is n-th Maclaurin coefficient of the function

$$z \to (1-z)^{-x-\alpha}(1-z)^{-x-\beta} - (1-z)^{-x}(1-z)^{-x-\alpha-\beta} \equiv 0 \quad \square$$

3. Main results. Introduce the notation

$$P_n(x) = Q_n^{1,1}(x-1) = \sum_{k=0}^{n} f_k f_{n-k} \binom{n}{k} \left[(x)_k (x)_{n-k} - (x+1)_k (x-1)_{n-k}\right]. \tag{7}$$

According to Lemma 3 $P_n(x) \equiv 0$ if $f_k = 1$ for all $k = 0, 1, \ldots, n$. Our main theorem is as follows.

Theorem 1 *If $f_k^2 > f_{k-1} f_{k+1}$ for $k = 1, 2, \ldots, n-1$, then $P_n(x)$ has degree $n-2$ and positive coefficients.*

Proof. Denote

$$\Phi_k(x) = 2(x)_k (x)_{n-k} - (x-1)_k (x+1)_{n-k} - (x-1)_{n-k}(x+1)_k \text{ for } k < n-k$$

and $\Phi_k(x) = (x)_k (x)_{n-k} - (x-1)_k (x+1)_{n-k}$ for $k = n-k$ (which only happens for even n). Then

$$P_n(x) = \sum_{0 \leq k \leq n/2} f_k f_{n-k} \binom{n}{k} \Phi_k(x).$$

Straightforward computation yields

$$\Phi_0(x) = -n(n-1)(x+1)_{n-2}, \tag{8}$$

$$\Phi_k(x) = (x)_{k-1}(x+1)_{n-k-2} l_k(x), \quad 1 \leq k \leq n/2, \tag{9}$$

where

$$l_k(x) = -A_k x + B_k, \tag{10}$$

$$A_k = n(n-1) - 4k(n-k), \quad B_k = n(n-1) - 2k(n-k), \quad 1 \leq k < n/2,$$

and

$$A_{n/2} = -n/2, \quad B_{n/2} = n(n-2)/4.$$

These formulas show that $\Phi_k(x)$ has degree $n-2$ for all $0 \le k \le n/2$ and the free term is only present in $\Phi_0(x)$, where it equals $-n!$, and in $\Phi_1(x)$, where it equals $(n-1)!$. Hence, the free term in $P_n(x)$ is equal to

$$-f_0 f_n \binom{n}{0} n! + f_1 f_{n-1} \binom{n}{1} (n-1)! = n!(f_1 f_{n-1} - f_0 f_n) > 0,$$

and it remains to prove the theorem for the coefficients at x^j for $j = 1, 2, \ldots, n-2$. Since for $n = 2$ we only have the free term we can assume that $n \ge 3$.

Now if $a_{k,j}$ is the coefficient at x^j, $j = 1, 2, \ldots, n-2$, in $\Phi_k(x)$, $k = 0, 1, \ldots, [n/2]$, then setting $M_{k,j} = \binom{n}{k} a_{k,j}$ we have according to Lemma 3:

$$\sum_{0 \le k \le n/2} M_{k,j} = 0, \quad j = 1, 2, \ldots, n-2.$$

Formula (8) shows that $a_{0,j} < 0$ for all $j = 1, 2, \ldots, n-2$. Hence, in order to apply Lemma 1 we only need to demonstrate that the sequence $a_{k,j}$, $k = 0, 1, \ldots, [n/2]$ has precisely one change of sign for each $j = 1, 2, \ldots, n-2$. We have

$$\Phi_1(x) = (n-1)(-(n-4)x + n - 2)(x+1)_{n-3}.$$

If $n = 3$ then this reduces to $2(x+1)$ and we are done, since the coefficient at x is positive and $[n/2] = 1$, so that $\Phi_1(x)$ is the last term. If $n = 4$ than $\Phi_1(x) = 6(x+1)$ and $\Phi_2(x) = 4x(x+1)$ which again proves the claim for $n = 4$. Hence, we may assume that $n \ge 5$.

Formula (5) and the definition of the Pochhammer symbol $(x)_m = x(x+1)\cdots(x+m-1)$ lead to representation

$$\Phi_k(x) = (x)_{k-1}(x+1)_{n-k-2} l_k(x) = x(-A_k x + B_k)(x+1)\cdots(x+k-2)(x+1)\cdots(x+n-k-2)$$

$$= x(-A_k x + B_k) \sum_{j=0}^{q} e_{q-j}(\chi_k) x^j = B_k e_q(\chi_k) x + \sum_{j=2}^{q+1} (B_k e_{q-j+1}(\chi_k) - A_k e_{q-j+2}(\chi_k)) x^j - A_k x^{q+2}$$

$$= B_k e_{p-1}(\chi_k) x + \sum_{j=2}^{p} (B_k e_{p-j}(\chi_k) - A_k e_{p-j+1}(\chi_k)) x^j - A_k x^{p+1},$$

where $2 \le k \le n/2$, $q = n-4$, $p = n-3$, $\chi_2 = \{1, 2, 3, \ldots, n-4\}$ and

$$\chi_k = \{1, 1, 2, 2, \ldots, k-2, k-2, k-1, k, k+1, \ldots, n-k-2\}, \quad k = 3, 4, \ldots \ldots$$

Note that each set χ_k, $k = 2, 3, \ldots \ldots$ has exactly $q = n-4$ elements. If $k = 1$ the formula is slightly different,

$$\Phi_1(x) = B_1 e_p(\chi_1) + \sum_{j=1}^{p} (B_1 e_{p-j}(\chi_1) - A_1 e_{p-j+1}(\chi_1)) x^j - A_1 x^{p+1}, \quad \text{with}$$

$$\chi_1 = \{1, 2, 3, \ldots, n-3\}.$$

The formula for $\Phi_k(x)$ shows that the coefficient at x is positive for all $k \ge 2$ since $B_k > 0$ for $0 \le k \le n/2$ by its definition. On the other hand, we know from (8) that the coefficient at x is negative for $k = 0$. Hence, irrespective of the sign of the coefficient at x for $k = 1$ our claim holds.

Thus we can narrow our attention to the coefficients at x^j for $j = 2, 3, \ldots, n-2$. Further, the coefficients at x^{n-2} are $-n(n-1), -A_1, -A_2, \ldots, -A_{[n/2]}$. We have $A_k = A(k)$ for

$$A(x) = n(n-1) - 4x(n-x).$$

Since $A(0) > 0$, $A(n/2) < 0$ and $A'(x) = 8x - 4n = 0$ at $x = n/2$, $A(x)$ is decreasing on $[0, n/2]$ and changes sign exactly once. So our claim is true for the coefficients at x^{n-2}.

Finally we need to handle the general case of the coefficients at x^j for $j = 2, 3, \ldots, n-3$. It is easy to that $\chi_{k-1} \prec^W \chi_k$ for $k = 3, 4, \ldots, [n/2]$ so that by Lemma 1

$$\frac{e_{p-j+1}(\chi_k)}{e_{p-j}(\chi_k)} < \frac{e_{p-j+1}(\chi_{k-1})}{e_{p-j}(\chi_{k-1})}$$

for $j = 2, 3, \ldots, n-3$ and $k = 3, 4, \ldots, [n/2]$. Further if $A_k < 0$ than it is clear that the coefficient at x^j is positive and there are no sign changes for such values of k. Hence we take those k for which $A_k \geq 0$. For such k the sequence B_k/A_k is increasing, since

$$\left(\frac{B(x)}{A(x)}\right)' = \frac{2n(n-1)(n-2x)}{A(x)^2} > 0, \quad B(x) = n(n-1) - 2x(n-x).$$

Now, if we assume that for some value of $k \in \{3, 4, \ldots, [n/2]\}$ the coefficient at x^j in $\Phi_k(x)$ is negative, i.e.

$$B_k e_{p-j}(\chi_k) - A_k e_{p-j+1}(\chi_k) < 0 \;\Leftrightarrow\; \frac{B_k}{A_k} < \frac{e_{p-j+1}(\chi_k)}{e_{p-j}(\chi_k)}.$$

Then for $k-1$ we will have

$$\frac{B_{k-1}}{A_{k-1}} < \frac{B_k}{A_k} < \frac{e_{p-j+1}(\chi_k)}{e_{p-j}(\chi_k)} < \frac{e_{p-j+1}(\chi_{k-1})}{e_{p-j}(\chi_{k-1})} \;\Leftrightarrow\; B_{k-1}e_{p-j}(\chi_{k-1}) - A_{k-1}e_{p-j+1}(\chi_{k-1}) < 0,$$

i.e. the coefficient at x^j is again negative in $\Phi_{k-1}(x)$. This proves that there can be no more than one change of sign in the sequence $\{a_{2,j}, a_{3,j}, \ldots, a_{[n/2],j}\}$ for each $j = 2, 3, \ldots, n-3$. It remains to consider $k = 2$. Introduce

$$\chi_2^{\varepsilon} = \{\varepsilon, 1, 2, \ldots, n-4\}.$$

Clearly, $\chi_1 \prec^W \chi_2^{\varepsilon}$ for each $0 < \varepsilon < 1$ and $e_m(\chi_2^{\varepsilon}) \to e_m(\chi_2)$ as $\varepsilon \to 0$ for $m = 0, 1, \ldots$. We have

$$B_2 e_{p-j}(\chi_2) - A_2 e_{p-j+1}(\chi_2) < 0 \;\Leftrightarrow\; \frac{B_2}{A_2} < \frac{e_{p-j+1}(\chi_2)}{e_{p-j}(\chi_2)} \;\Rightarrow\; \frac{B_2}{A_2} < \frac{e_{p-j+1}(\chi_2^{\varepsilon})}{e_{p-j}(\chi_2^{\varepsilon})}$$

for sufficiently small $\varepsilon > 0$ and

$$\frac{B_1}{A_1} < \frac{B_2}{A_2} < \frac{e_{p-j+1}(\chi_2^{\varepsilon})}{e_{p-j}(\chi_2^{\varepsilon})} < \frac{e_{p-j+1}(\chi_1)}{e_{p-j}(\chi_1)}. \;\square$$

4. Conjectures for higher order determinants. For $f(x; z)$ defined in (2) let us consider the Toeplitz determinant

$$F_r(x, z) = \begin{vmatrix} f(x; z) & f(x+1; z) & f(x+2; z) & \cdots & f(x+r-1; z) \\ f(x-1; z) & f(x; z) & f(x+1; z) & \cdots & f(x+r-2; z) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f(x-r+1; z) & f(x-r+2; z) & f(x-r+3; z) & \cdots & f(x; z) \end{vmatrix}.$$

Compute

$$F_r(x,z) = \begin{vmatrix} \sum\limits_{k_1=0}^{\infty} f_{k_1}(x)_{k_1}\frac{z^{k_1}}{k_1!} & \sum\limits_{k_1=0}^{\infty} f_{k_1}(x+1)_{k_1}\frac{z^{k_1}}{k_1!} & \cdots & \sum\limits_{k_1=0}^{\infty} f_{k_1}(x+r-1)_{k_1}\frac{z^{k_1}}{k_1!} \\ \sum\limits_{k_2=0}^{\infty} f_{k_2}(x-1)_{k_2}\frac{z^{k_2}}{k_2!} & \sum\limits_{k_2=0}^{\infty} f_{k_2}(x)_{k_2}\frac{z^{k_2}}{k_2!} & \cdots & \sum\limits_{k_2=0}^{\infty} f_{k_2}(x+r-2)_{k_2}\frac{z^{k_2}}{k_2!} \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k_r=0}^{\infty} f_{k_r}(x-r+1)_{k_r}\frac{z^{k_r}}{k_r!} & \sum\limits_{k_r=0}^{\infty} f_{k_r}(x-r+2)_{k_r}\frac{z^{k_r}}{k_r!} & \cdots & \sum\limits_{k_r=0}^{\infty} f_{k_r}(x)_{k_r}\frac{z^{k_r}}{k_r!} \end{vmatrix}$$

$$= \sum_{k_1,k_2,\dots,k_r=0}^{\infty} f_{k_1}f_{k_2}\cdots f_{k_r}\frac{z^{k_1+k_2+\cdots+k_r}}{k_1!k_2!\cdots k_r!}\begin{vmatrix} (x)_{k_1} & (x+1)_{k_1} & \cdots & (x+r-1)_{k_1} \\ (x-1)_{k_2} & (x)_{k_2} & \cdots & (x+r-2)_{k_2} \\ \vdots & \vdots & \cdots & \vdots \\ (x-r+1)_{k_r} & (x-r+2)_{k_r} & \cdots & (x)_{k_r} \end{vmatrix}$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k_1+k_2+\cdots+k_r=n}\binom{n}{k_1,k_2,\dots,k_r} f_{k_1}f_{k_2}\cdots f_{k_r}\begin{vmatrix} (x)_{k_1} & (x+1)_{k_1} & \cdots & (x+r-1)_{k_1} \\ (x-1)_{k_2} & (x)_{k_2} & \cdots & (x+r-2)_{k_2} \\ \vdots & \vdots & \cdots & \vdots \\ (x-r+1)_{k_r} & (x-r+2)_{k_r} & \cdots & (x)_{k_r} \end{vmatrix}.$$

Hence,

$$F_r(x,z) = \sum_{n=0}^{\infty} z^n P_n^r(x),$$

where

$$P_n^r(x) := \sum_{k_1+k_2+\cdots+k_r=n}\binom{n}{k_1,k_2,\dots,k_r} f_{k_1}f_{k_2}\cdots f_{k_r}\begin{vmatrix} (x)_{k_1} & (x+1)_{k_1} & \cdots & (x+r-1)_{k_1} \\ (x-1)_{k_2} & (x)_{k_2} & \cdots & (x+r-2)_{k_2} \\ \vdots & \vdots & \cdots & \vdots \\ (x-r+1)_{k_r} & (x-r+2)_{k_r} & \cdots & (x)_{k_r} \end{vmatrix}.$$

Of course, $P_n^2(x) = P_n(x) = Q_n^{1,1}(x-1)$. To conjecture a reasonable generalization of Theorem 1 we need to recall the notion of the Pólya frequency sequences, first introduced by Fekete in 1912. They were studied in detail by Karlin in [Karlin]. The class of all Pólya frequency sequences of order $1 \le r \le \infty$ is denoted by PF_r and consists of the sequences $\{f_k\}_{k=0}^{\infty}$ such that all minors of order $\le r$ (all minors if $r = \infty$) of the infinite matrix

$$\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots \\ 0 & f_0 & f_1 & f_2 & \cdots \\ 0 & 0 & f_0 & f_1 & \cdots \\ 0 & 0 & 0 & f_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

are non-negative. Clearly, $PF_1 \supset PF_2 \supset \cdots \supset PF_\infty$. The PF_2 sequences are precisely the log-concave sequences without internal zeros. Our conjectures are

Conjecture 4 *Suppose $\{f_k\}_{k=0}^{n} \in PF_r$, $r \ge 2$. Then the polynomial $P_n^r(x)$ has degree $n-r(r-1)$ and positive coefficients.*

Conjecture 5 *Suppose $\{f_k\}_{k=0}^{n} \in PF_r$. Then the polynomial $P_n^r(x)$ is Hurwitz stable.*

Conjecture 6 *Suppose $\{f_k\}_{k=0}^n \in PF_\infty$. Then all zeros of the polynomial $P_n^r(x)$ are real and negative for each $r \geq 2$.*

Again, Conjecture 4 follows from Conjecture 5 but both are independent of Conjecture 6.

Conjectures 3 and 6 bear certain resemblance to the recent research of Brändén [Br], Grabarek [Gr] and Yoshida [Yo]. Among other things, these works consider non-linear operators on polynomials that preserve the class of polynomials with real negative zeros. According to the celebrated theorem of Aissen, Schoenberg and Whitney [ASW] the sequence $\{f_0, f_1, \ldots, f_n\}$ is a PF_∞ sequence iff $\sum_{k=0}^n f_k x^k$ has only real negative zeros. In particular, Brändén found necessary and sufficient conditions on the real sequence α_j to ensure that the operators

$$\{f_k\}_{k=0}^n \to \left\{ \sum_{j=0}^{\infty} \alpha_j f_{m-j} f_{m+j} \right\}_{m=0}^{n} \quad \text{and} \quad \{f_k\}_{k=0}^n \to \left\{ \sum_{j=0}^{\infty} \alpha_j f_{m-j} f_{m+1+j} \right\}_{m=0}^{n-1} \tag{12}$$

preserve PF_∞. Here $f_i = 0$ if $i \notin \{0, 1, \ldots, n\}$. Using Brändén's criterion Grabarek showed in [Gr] that the transformation ($p > 0$ is an integer)

$$\{f_k\}_{k=0}^n \to \left\{ \binom{2p-1}{p} f_m^2 + \sum_{j=1}^{p} (-1)^j \binom{2p}{p-j} f_{m-j} f_{m+j} \right\}_{m=0}^{n} \tag{13}$$

preserves PF_∞. Conjectures 3 and 6 also assert that certain non-linear transformations preserve PF_∞. For $r = 2$ this transformation is easy to write explicitly. Denote by $p_n(m)$, $m = 0, 1, \ldots, n-2$, the coefficient at x^m of the polynomial $P_n(x)$. Then

$$p_n(0) = n!(f_1 f_{n-1} - f_0 f_n), \tag{14}$$

and

$$p_n(m) = \frac{1}{m!} \sum_{0 \leq k \leq n/2} f_k f_{n-k} \binom{n}{k} \frac{d^m}{dx^m} \Phi_k(x)_{|x=0}, \quad m = 1, 2, \ldots, n-2.$$

For $k = 0$ we have

$$[\Phi_0(x)]_{|x=0}^{(m)} = -n(n-1)[(x+1)_{n-2}]_{|x=0}^{(m)} = -n(n-1) \left[\sum_{j=0}^{n-2} S_{j+1}^{n-1} x^j \right]_{|x=0}^{(m)} = -n(n-1) m! S_{m+1}^{n-1},$$

where S_j^p is unsigned Stirling numbers of the first kind that can be defined by $(x)_p = \sum_{j=1}^p S_j^p x^j$. For $k = 1$ we obtain,

$$[\Phi_1(x)]_{|x=0}^{(m)} = [(x+1)_{n-3} l_1(x)]_{|x=0}^{(m)} = B_1 [(x+1)_{n-3}]_{|x=0}^{(m)} - A_1 m [(x+1)_{n-3}]_{|x=0}^{(m-1)} = m!(B_1 S_{m+1}^{n-2} - A_1 S_m^{n-2}),$$

and for $2 \leq k \leq n/2$, compute

$$[\Phi_k(x)]_{|x=0}^{(m)} = [(x)_{k-1}(x+1)_{n-k-2} l_k(x)]_{|x=0}^{(m)} = B_k [(x)_{k-1}(x+1)_{n-k-2}]_{|x=0}^{(m)} - A_k m [(x)_{k-1}(x+1)_{n-k-2}]_{|x=0}^{(m-1)},$$

$$[(x)_{k-1}(x+1)_{n-k-2}]_{|x=0}^{(m)} = \sum_{i=0}^{m} \binom{m}{i} \left[\sum_{j=1}^{k-1} S_j^{k-1} x^j \right]_{|x=0}^{(i)} \left[\sum_{j=0}^{n-k-2} S_{j+1}^{n-k-1} x^j \right]_{|x=0}^{(m-i)}$$

$$= m! \sum_{i=1}^{m} S_i^{k-1} S_{m-i+1}^{n-k-1}, \quad 2 \leq k \leq n/2.$$

Hence,

$$[\Phi_k(x)]^{(m)}_{|x=0} = B_k m! \sum_{i=1}^{m} S_i^{k-1} S_{m-i+1}^{n-k-1} - A_k m! \sum_{i=1}^{m-1} S_i^{k-1} S_{m-i}^{n-k-1}, \quad 2 \le k \le n/2.$$

Finally, we get for $m = 1, 2, \dots, n-2$,

$$\begin{aligned} p_n(m) &= -n(n-1) S_{m+1}^{n-1} f_0 f_n + f_1 f_{n-1} n (B_1 S_{m+1}^{n-2} - A_1 S_m^{n-2}) \\ &\quad + \sum_{2 \le k \le n/2} f_k f_{n-k} \binom{n}{k} \left(B_k \sum_{i=1}^{m} S_i^{k-1} S_{m-i+1}^{n-k-1} - A_k \sum_{i=1}^{m-1} S_i^{k-1} S_{m-i}^{n-k-1} \right) \\ &= -n(n-1) S_{m+1}^{n-1} f_0 f_n + \sum_{1 \le k \le n/2} f_k f_{n-k} \binom{n}{k} \sum_{i=0}^{m} S_i^{k-1} \left(B_k S_{m-i+1}^{n-k-1} - A_k S_{m-i}^{n-k-1} \right), \end{aligned} \tag{15}$$

where

$$S_q^p = 0, \quad q > p, \quad S_0^p = 0, \quad p \ge 1, \quad S_0^0 = 1.$$

So Conjecture 3 can be restated as the assertion that the non-linear operator

$$\{f_k\}_{k=0}^{n} \to \left\{ \sum_{0 \le j \le n/2} f_j f_{n-j} P_{j,m} \right\}_{m=0}^{n-2},$$

where the numbers $P_{j,m}$ can be read off (14) and (15), preserves PF_∞. Both Brändén's transformation (12) our transformation above are bilinear forms but of somewhat different character. One may ask then what conditions on the numbers $P_{k,m}$ would ensure the preservation of PF_∞.

5. Some remarks on numerical experiments. In order to run numerical experiments with Conjectures 1 to 6 one has to be able to generate PF_r sequences. For $r = 2$ and $r = \infty$ the methods are quite clear. Setting

$$\delta_k = \frac{f_k^2}{f_{k-1} f_{k+1}}$$

we obtain for $\{f_k\}_{k=0}^{\infty} \in PF_2$:

$$f_k = f_0^{k+1} \delta_1^k \delta_2^{k-1} \cdots \delta_k, \tag{16}$$

where $f_0 > 0$ and $0 < \delta_j \le 1$, $j = 1, 2, \dots, n$, $\delta_j = 0$, $j > n$. Hence, we can parameterize all PF_2 sequences by sequences with elements from $(0, 1]$. Generating the latter randomly we get a random PF_2 sequence. Next, for $r = \infty$ we can simply generate n random positive numbers $a_1, a_2, \dots, a_n$ and compute the coefficients of the polynomial $\prod_{i=1}^{n} (x + a_i)$ producing by Aissen-Shoenberg-Whitney theorem a PF_∞ sequence. According to the same theorem all finite PF_∞ sequences are obtained in this way.

The situation is less clear for $3 \le r < \infty$. I am unaware of any method to parameterize all PF_r sequences for these values of r. However, some subclasses can be parameterized. One possible method is provided by the following result of Katvova and Vishnyakova [KV, Corollary of Theorem 5]: if nonnegative sequence $\{f_n\}_{n=0}^{\infty}$ satisfies

$$f_n^2 \ge 4 \cos^2 \left(\frac{\pi}{r+1} \right) f_{n-1} f_{n+1}, \quad n \ge 1,$$

then $\{f_n\}_{n=0}^{\infty} \in PF_r$. This implies that if we choose $0 < \delta_j \le \left(4 \cos^2 \frac{\pi}{r+1} \right)^{-1}$ then the sequence generated by (16) is a PF_r sequence. Another method to produce a finite PF_r sequence follows

from Shoenberg's theorem [Sch] stating that the coefficients of a polynomial with zeros lying in the sector $|\arg z - \pi| < \pi/(r+1)$ form a PF_r sequence. Hence, generating such zeros randomly and doubling their number by adding the complex conjugate to each we get a polynomial with PF_r coefficients.

Finally, Ostrovskii and Zheltukhina [OZ] parameterized a large subclass of PF_3 sequences. Namely, a PF_3 sequence $\{f_0, f_1, f_2, \ldots,\}$ is Q_3 if all truncated sequences $\{f_i\}_{i=0}^n$ are also PF_3 for each $n = 1, 2, \ldots$. The main Theorem of [OZ] states that a sequence $\{f_0, f_1, f_2, \ldots,\}$ is Q_3 iff $f_0 > 0$, $f_1 = f_0\beta \geq 0$ and

$$f_n = \frac{f_0\beta^n\delta_2^{n-1}\delta_3^{n-2}\cdots\delta_{n-1}^2\delta_n}{\alpha_2^{n/2}\alpha_3^{(n-1)/2}\alpha_4^{(n-2)/2}\cdots\alpha_{n-1}^{3/2}\alpha_n},$$

where

$$\alpha_2 = 1+\delta_2,\ \alpha_3 = 1+\delta_3\sqrt{\alpha_2},\ \alpha_4 = 1+\delta_4\sqrt{\alpha_3}, \ldots, \quad 0 \leq \delta_j \leq 1, \quad j = 2, 3, \ldots$$

and the sequence $\{\delta_j\}$ has no internal zeros. This theorem provides a simple method of generating random Q_3 sequences.

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